Faculty of Civil Engineering Belgrade
Master Study COMPUTATIONAL ENGINEERING
Numerical Methods
Fall semester 2005/2006

Midterm exam

[**1**] For given (integer) matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

find the inverse, using Gauss-Jordan method.
Hint: Proceed transformation

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{c|c} \quad \mathbf{I} \quad & \quad \mathbf{A}^{-1} \quad \end{array}\right].$$

[**2**] Draw the algorithm for solution of system of equation

$$\begin{bmatrix} 10 & 3 & -1 \\ -1 & 5 & -1 \\ 1 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 13 \end{bmatrix}.$$

using Jacobi method with exactness $\varepsilon = 10^{-3}$ and proceed the first iteration (Take $\vec{x}^{(0)} = \vec{b}$).

[**3**] Draw the algorithm for finding the real root of equation

$$x^3 - x - 1 = 0$$

by iteration method, with $\varepsilon = 10^{-3}$. For starting value of x take $x_0 = 1.5$.

[4] Draw the algorithm for root finding of equation

$$f(x) = e^{-x} - x = 0$$

by Newton method with $\varepsilon = 5 \cdot 10^{-2}$. Starting value of x determine graphically.

[5] In the year 1225 explored Leonardo of Pisa the equation

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0$$

and got the root $x = 1.368808107$, using unknown method. This result was outstanding for that time.

Draw the algorithm and solve the equation of Leonardo by reduction to the form $x = F(x)$ (fixed point method).

Hint: Take

$$x = F(x) = \frac{20}{x^2 + 2x + 10}, \text{ i.e.}$$
$$x_n = \frac{20}{x_{n-1}^2 + 2x_{n-1} + 10}, \text{ and } x_0 = 1.$$

Take $\varepsilon = 10^{-3}$.

[6] Approximate function $x \mapsto f(x) = e^x$ on interval $[0, 0.5]$ by interpolating polynomial (use Lagrange's interpolation formula).

Hint: Function e^x given in tabular form

k	0	1	2
x_k	0.0	0.2	0.5
$f(x_k)$	1.00000	1.221403	1.648721

The approximative polynomial is of form $P_n(x) = a_0x^n + a_1x_{n-1} +$

$\cdots + a_n$, given in Lagrange's form

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x),$$

$$L_k = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

$$= \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i},$$

i.e.

$$P_n(x) = \sum_{k=0}^{n} f(x_k) \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i}.$$

[7] For given table of values $y(x) = \sqrt{x}$ form a table of differences up to Δ^6. Apply the table to calculate $\sqrt{1.005}$ with $n = 1$ (linear approximation) using Newton's forward differences.

k	x_k	$y(x) = \sqrt{x}$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	1.0	1.0000						
			50					
1	1.01	1.0050		0				
			50		−1			
2	1.02	1.0100		−1		2		
			49		1		−3	
3	1.03	1.0149		0		−1		4
			49		0		1	
4	1.04	1.0198		0		0		
			49		0			
5	1.05	1.0247		0				
			49					
6	1.06	1.0296						

Hint: The Newton's formula is

$$P_k = y_0 + \binom{k}{1}\Delta y_0 + \frac{k}{2}\binom{k}{2}\Delta^2 y_0 + \cdots + \binom{k}{n}\Delta^n y_0,$$

[8] Using least-square (discrete) method determine parameters a_0 and a_1 in approximate function $\Phi(x) = a_0 + a_1 x$ for the following set of data:

j	0	1	2	3
x_j	0	1	2	4
$f(x_j)$	1	3	0	-1

Hint: Use the matrix equation

$$\mathbf{X}^\mathbf{T} \cdot \mathbf{X} \cdot \vec{\mathbf{a}} = \mathbf{X}^\mathbf{T} \cdot \vec{\mathbf{f}},$$

where, if we use the basic functions $\phi_i(x) = x^i \quad (i = 0, 1, \ldots, n)$,

$$\mathbf{X} = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & & & \\ 1 & x_m & x_m^2 & \ldots & x_m^n \end{bmatrix}, \qquad (m \geq n).$$

[9] Proceed the discrete least squares approximation on the data set

x_i	1.1	1.9	4.2	6.1
$f(x_i)$	2.5	3.2	4.5	6.0

with function $\Phi_0(x) = a_0 + a_1 x$.

[10] Draw an algorithm for applying the Simpson's integration rule to compute

$$\int_0^{\pi/2} \sin x \, dx$$

taking $h = \pi/8$ and halving it up to $\pi/2048$. Compare the results with exact one.

[11] Compute the integral of error-function

$$H(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

for $x = 0.5$ and $x = 1$, using Taylor's series, with exactness $\varepsilon = 10^{-3}$.

Hint: Use series

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \frac{t^8}{24} - \frac{t^{10}}{120} + \cdots.$$

[12] Use Newton-Cotes' formula (trapezoidal) to compute the integral

$$I = \int_a^b f(x)\, dx$$

of function given in tabular form

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	1.0000	0.8333	0.7143	0.6250	0.5556	0.5000

with $a = 1,\ b = 2$. Compare the results with exact one, i.e.

$$\int_1^2 \frac{1}{x}\, dx = ln\ x\ \Big|_1^2 = ln\ 2 = 0.6931$$

Draw the algorithm for computation of arbitrary integral by trapezoidal rule.