Faculty of Civil Engineering Belgrade
Master Study COMPUTATIONAL ENGINEERING
Fall semester 2005/2006

ASSIGNMENTS

LESSONS I-III

[**1**] Carry out Gauss elimination to solve the system:

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & 2 & 3 & -1 \\ 4 & 5 & -9 & 6 \\ 1 & 1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 28 \\ 7 \end{bmatrix}.$$

[**2**] Solve the following system in three different ways (arbitrary):

$$\begin{bmatrix} 0.31 & 0.14 & 0.30 & 0.27 \\ 0.26 & 0.32 & 0.18 & 0.24 \\ 0.61 & 0.22 & 0.20 & 0.31 \\ 0.40 & 0.34 & 0.36 & 0.17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.02 \\ 1.00 \\ 1.34 \\ 1.27 \end{bmatrix}.$$

Write a program subroutines in an arbitrary programming language.

[**3**] Solve the following system by Gauss elimination

$$\begin{bmatrix} 2.304 & -1.213 & 2.441 \\ 8.752 & -5.608 & 3.916 \\ 1.527 & 4.333 & -2.214 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7.201 \\ 9.284 \\ 3.551 \end{bmatrix}.$$

[**4**] Solve the following system

$$\begin{bmatrix} 0.000003 & 0.213472 & 0.332147 \\ 0.215512 & 0.375623 & 0.476625 \\ 0.173257 & 0.663257 & 0.625675 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.235262 \\ 0.127653 \\ 0.285321 \end{bmatrix}.$$

a) using Gauss elimination,
b) using Gauss elimination with pivoting.
Compare results (use six digits).

[**5**] For given (integer) matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

find the inverse, using
a) Gauss elimination,
b) Gauss-Jordan method.
Hint: Proceed transformation

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 6 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{c|c} \quad \mathbf{I} \quad & \quad \mathbf{A}^{-1} \quad \end{array}\right].$$

[**6**] Explore the convergence of iterative solution of system of linear equations

$$\begin{bmatrix} 10 & 3 & -1 \\ -1 & 5 & -1 \\ 1 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 13 \end{bmatrix}.$$

and proceed solution using
a) Jacobi method,
b) Gauss-Seidel Method.
(Solution: $\vec{\mathbf{x}} = [1\ 1\ 1]^T$.

[**7**] Solve the system

$$\begin{bmatrix} 3 & 4 & -1 \\ 2 & 6 & 3 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix}.$$

using

a) Jacobi method,

b) Gauss-Seidel Method.

[8] There are three widely held errors about iteration. The following three statements correct these errors:

A. *Diagonal dominance is not required for either Jacobi or Gauss-Seidel method.*

Example: find the $\mathbf{A}^{-1}$ using iterative method for given

$$\mathbf{A} = \begin{bmatrix} 8 & 2 & 1 \\ 10 & 4 & 1 \\ 50 & 25 & 2 \end{bmatrix},$$

with $\varepsilon = 0.1$. Find the inverse, using also

a) Gauss elimination,

b) Gauss-Jordan method.

Compare with exact solution yield with Mathematica.

B. *It is not always true that if the Jacobi method works, then Gauss-Seidel works even better.*

The belief that Gauss-Seidel is always better than Jacobi comes from the fact that if $\mathbf{A}$ is symmetric and positive definite and satisfied certain other conditions, then Gauss-Seidel converges as fast as Jacobi.

C. *Iteration methods are not better for ill-conditioned problems than Gauss elimination.*

It is often believed that the source of trouble in Gauss elimination for ill-conditioned problems is the steady accumulation of round-off errors during the computation. This is not so – the inaccuracy often arises almost entirely in one single arithmetic step of elimination.

[9] Write a program to use Gauss-Seidel iteration to solve the system

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_{i-1} + 2x_i - x_{i+1} &= 0 \quad (i = 2, 3, \ldots, n-1) \\ -x_{n-1} + 2x_n &= 1, \end{aligned}$$

for $n = 20$. Start with $\mathbf{x^{(0)}} = \mathbf{0}$ and terminate the iteration when

$$\frac{||\mathbf{x^{(k)}} - \mathbf{x^{(k-1)}}||_\infty}{||\mathbf{x^{(k)}}||} < 10^{-6}.$$

(The exact solution is $x_i = 1$ for all i.) What is the accuracy achieved? Compare the computer time required for Gauss-Seidel and Gauss elimination.

[**10**] Apply the Jacobi method to previous problem for 100 iterations and find the number of Gauss-Seidel iterations that gives the same accuracy.

[**11**] Solve the system

$$\begin{bmatrix} 2 & 1 & -1/2 & 8 \\ 6 & 2 & 8 & 14 \\ 1 & -6 & 10 & 9 \\ 2 & 11 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 1 \\ 4 \end{bmatrix}.$$

by Gauss-Seidel iteration. Take $\mathbf{x^{(0)}} = [0\ 0\ 0\ 0]^T$.

[**12**] Apply Jacobi and Gauss-Seidel iteration to the problem $\mathbf{A}\vec{\mathbf{x}} = [8\ 8\ 29]^T$ where

$$\mathbf{A} = \begin{bmatrix} 8 & 2 & 1 \\ 10 & 4 & 1 \\ 50 & 25 & 2 \end{bmatrix},$$

with the initial guess $\mathbf{x^{(0)}} = [-1\ 1\ 0]^T$. How many iterations does it take for these to converge to 10^{-4} accuracy?

[**13**] Solve the system $\mathbf{A}\vec{x} = \vec{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 20 & -400 \\ 0.2 & -2 & -20 \\ -0.04 & -0.2 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0.2 \\ 0.05 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

by Gauss algorithm.

[**14**] Solve the following system $\mathbf{A}\vec{x} = \vec{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

by Gauss algorithm with pivoting.

[**15**] Consider the linear equation $\mathbf{A}\vec{x} = \vec{b}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & -4 \\ 6 & 9 & a \end{bmatrix}, \quad a \in \Re, \quad \vec{b} = \begin{bmatrix} 12 \\ -24 \\ -6 \end{bmatrix}.$$

Perform the **LU**-decomposition, compute the determinant of $\mathbf{A}$, and decide for which values of a the matrix is nonsingular. Provide the solution for $a = 3$.

[**16**] Find the inverse matrix of regular matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix},$$

using Gauss algorithm.

[**17**] For given matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -1 & 2 & -1 \\ 3 & 14 & 4 & 1 \\ 1 & 2 & 2 & 9 \end{bmatrix}$$

find the factorization $A = LU$, where L lower triangle and U upper triangle with unit diagonal. Using factorization solve the system $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{b} = [9\ 0\ 22\ 14]^T$.

[**18**] For tridiagonal matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 8 & 5 & -2 & 0 & 0 \\ 0 & -3 & -1 & 5 & 0 \\ 0 & 0 & -9 & 13 & -4 \\ 0 & 0 & 0 & -2 & -3 \end{bmatrix}$$

find LU factorization with unit diagonal in L and then find the solutions of $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{\mathbf{b}} = [1\ 0\ 1\ 0\ 1]^T$.
Hint: If given tridiagonal matrix

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & \dots & 0 \\ a_1 & b_2 & c_2 & & 0 \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \dots & 0 & a_{n-1} & b_n \end{bmatrix}$$

factorizes in two matrices of form

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \alpha_2 & 1 & 0 & & 0 & 0 \\ 0 & \alpha_3 & 1 & & 0 & 0 \\ \vdots & & & & & \\ 0 & \dots & 0 & & \alpha_n & 1 \end{bmatrix}$$

and

$$\mathbf{U} = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \gamma_2 & & 0 & 0 \\ 0 & 0 & \beta_3 & & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & 0 & \beta_n \end{bmatrix} \qquad (\beta_1\beta_2 \dots \beta_n \neq 0)$$

By comparing corresponding elements of matrix A and matrix

$$\mathbf{LU} = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ \alpha_2\beta_1 & \alpha_2\gamma_1+\beta_2 & \gamma_2 & & 0 & 0 \\ 0 & \alpha_3\beta_2 & \alpha_3\gamma_2+\beta_3 & & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & \alpha_n\beta_{n-1} & \alpha_n\gamma_{n-1}+\beta_n \end{bmatrix},$$

we get the following recursive formulas for determination of elements $\alpha_i, \beta_i, \gamma_i$:

$$\begin{array}{ll} \beta_1 = b_1, & \gamma_{i-1} = c_{i-1}, \\ \alpha_i = \frac{a_i}{\beta_{i-1}}, & \beta_i = b_i - \alpha_i\gamma_{i-1}, \qquad (i = 2, 3, \dots, n), \end{array}$$

[**19**] In the numerical treatment of ordinary differential equations one encounters the tridiagonal $n \times n$-matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

Perform a few steps of **LU** decomposition and guess the general formula for the coefficient of **L** and **U**. Repeat the same for the (symmetric) $\mathbf{LDL^T}$ decomposition.

Hint: See the solution of previous problem.

[**20**] For given

$$\mathbf{A} = \begin{bmatrix} -3 & 5 & -11 & -13 \\ 2 & -1 & 4 & 7 \\ 6 & -6 & 12 & 24 \\ 3 & 1 & -2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 9 \\ -2 \\ -6 \\ 5 \end{bmatrix},$$

by Gauss algorithm with pivoting define the permutation matrix P and triangle matrices L and U in the factorization $LR = PA$. Find the solution of system $\mathbf{A}\vec{x} = \vec{b}$ using obtained factorization.

[21] Solve the following system

$$10x_1 + 3x_2 - x_3 = 12$$
$$-x_1 + 5x_2 - x_3 = 3$$
$$x_1 + 2x_2 + 10x_3 = 13$$

by Jacobi iterative method.

[22] Given a system of linear equations

$$\begin{array}{ccccccccc} 5x_1 & - & x_2 & + & x_3 & + & 3x_4 & = & 2 \\ & & 5x_2 & + & 2x_3 & - & x_4 & = & 0 \\ x_1 & - & 2x_2 & + & 3x_3 & + & x_4 & = & 4 \\ x_1 & - & x_2 & + & 3x_3 & + & 4x_4 & = & 10 \end{array}$$

Form the Gauss-Seidel method and explore its convergency.

[23] Solve the following set of equations by use of determinants (Cramer's rule):

$$\begin{array}{ccccccc} 1.4x_1 & + & 2.3x_2 & + & 3.7x_3 & = & 6.5 \\ 3.3x_1 & + & 1.6x_2 & + & 4.3x_3 & = & 10.3\,. \\ 2.5x_1 & + & 1.9x_2 & + & 4.1x_3 & = & 8.8 \end{array}$$

[24] Solve the previous set of equations by Gauss reduction.

[25] Show that the equations

$$\begin{array}{ccccccc} \omega x_1 & + & 3x_2 & + & x_3 & = & 5 \\ 2x_1 & - & x_2 & + & 2\omega x_3 & = & 3 \\ x_1 & + & 4x_2 & + & \omega x_3 & = & 6 \end{array}$$

posses a unique solution when $\omega \neq \pm 1$, that no solution exists when $\omega = -1$, and that infinitely many solutions exist when $\omega = 1$. In addition, investigate the corresponding situation when the right-hand members are replaced by zeros.

[**26**] Solve the following set of equations

$$\begin{bmatrix} 8.467 & 5.137 & 3.141 & 2.063 \\ 5.137 & 6.421 & 2.617 & 2.003 \\ 3.141 & 2.617 & 4.128 & 1.628 \\ 2.063 & 2.003 & 1.628 & 3.446 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 29.912 \\ 25.058 \\ 16.557 \\ 12.690 \end{bmatrix}.$$

and find $\mathbf{A}^{-1}$ by

a) Gauss reduction,

b) Gauss-Jordan reduction.

[**27**] Solve the equations

$$\begin{array}{ccccccccc} 10.01x_1 & + & 6.99x_2 & + & 8.01x_3 & + & 6.99x_4 & = & 32 \\ 6.99x_1 & + & 5.01x_2 & + & 5.99x_3 & + & 5.01x_4 & = & 23 \\ 8.01x_1 & + & 5.99x_2 & + & 10.01x_3 & + & 8.99x_4 & = & 33 \\ 6.99x_1 & + & 5.01x_2 & + & 8.99x_3 & + & 10.01x_4 & = & 31 \end{array}$$

by arbitrary direct and an arbitrary iterative method.

[**28**] Using direct methods for solution of system of linear equations

a) Gauss reduction,

b) Gauss-Jordan method,

c) **LU** decomposition,

and iterative methods

a) Jacobi method,

b) Gauss-Seidel method, compute the forces in members of statically determined plane grid given on the next figure.

Remark: Compare the results. Perform control computation using program STRESS, SAP or similar one.

[**29**] *Solve the system of linear equations by Cramer's rule supposing that each determinant is evaluated by the standard scheme (expansion by minors) as the sum of $(n-1)!$ products of n factors each. Estimate how many multiplications are then required to solve $\mathbf{A}\vec{x} = \vec{b}$ for $n = 10, 25, 30, 50, 100$. Translate this into the computer time if each multiplication (and all associated other operations) takes $1\mu sec$. For large n use Sterling's formula

$$n! \sim \sqrt{2\pi n}\exp^{-n} n^n.$$

(There are about $3 \times 10^{15}\mu sec$ in a century. The sun will burned out before the computation with $n = 100$ is complete).

[**30**] * An "x-matrix" has a pattern of nonzero elements that forms an "x" as the following 5×5 and 6×6 examples.

$$\begin{bmatrix} X & X & X & X & X \\ & X & X & X & \\ & & X & & \\ & X & X & X & \\ X & X & X & X & X \end{bmatrix} \begin{bmatrix} X & X & X & X & X & X \\ & X & X & X & X & \\ & & X & X & & \\ & & X & X & & \\ & X & X & X & X & \\ X & X & X & X & X & X \end{bmatrix}$$

a) In the spirit of Gauss elimination describe an algorithm to transform a general matrix into this form by solving a sequence 2 by 2 problems.

b) Show how one can back substitute in an x-matrix by solving a sequence of 2×2 problems.

c) Make operations counts for parts a) and b) and compare the work of this scheme with ordinary Gauss elimination. Ignore pivoting.

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* Problem could be used as basis for semester research paper

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[**31**] * Consider a system of equations that "almost" breacks up into two parts:

$$\sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{m} e_{ij}y_j = c_i \quad i = 1, 2, \ldots, n$$
$$\sum_{j=1}^{n} h_{ij}x_j + \sum_{j=1}^{m} b_{ij}y_j = c_{i+n} \quad i = 1, 2, \ldots, m.$$

The matrices E and H are "small" compared to A and B. Suppose that A is much smaller than B and that B is heavily diagonally dominant. So one proceeds to apply the following scheme:

step 1 Guess at $\mathbf{y}$ and solve for $\mathbf{x}$ exactly from the first set of equations.

step 2 Given this $\mathbf{x}$, solve for an improved estimate of $\mathbf{y}$ by one iteration of Gauss-Seidel on the second set of equations.

Then iterate steps 1 and 2 to take a vector pair $(\mathbf{x}^{(k)}, \mathbf{y}^{(k)}$ into $(\mathbf{x}^{(k+1)}, \mathbf{y}^{(k+1)}$.

a) Express the original linear system in matrix form;
b) Express steps 1 and 2 in matrix form;
c) Express the iteration scheme in matrix form;
d) Explore the matrix which governs the convergence of this scheme.

Make a constructional model of this mathematical model. Make a dynamical analysis of the system (eigenvalues). Analyze the influence of matrices E and H on the eigenfrequence of the system.