Faculty of Civil Engineering Belgrade
Master Study COMPUTATIONAL ENGINEERING
Fall semester 2005/2006

ASSIGNMENTS

LESSON VIII

Approximations of Functions

[**1**] Approximate function $x \to f(x) = \cos x$ by function $x \to \Phi(x) = a_0 + a_1 x$ in space

a. $L^1(0, \pi/2)$,

b. $L^2(0, \pi/2)$.

Hint: a) Minimize a norm

$$J(a_0, a_1) = ||\delta_1||_1 = \int_0^{\pi/2} |\cos x - a_0 - a_1 x| \; dx,$$

$$\frac{\partial J}{\partial a_0} = \frac{\partial}{\partial a_0}[\int_0^{\pi/2} |\cos x - a_0 - a_1 x| \; dx] = \frac{\partial}{\partial a_0}[\sin x - a_0 x - \frac{a_1 x^2}{2}] = 0,$$

$$\frac{\partial J}{\partial a_1} = \frac{\partial}{\partial a_1}[(-x)|\cos x - a_0 - a_1 x|] = 0.$$

[**2**] Find the least-square approximation to function $x \to f(x) = \sin x$ on segment $[-\pi, \pi]$ with weight $x \to p(x) = 1$ and polynomial of degree not greater than three.

Hint: Take $\Phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Due to fact that function $\sin x$ is odd and symmetry of segment on the area of approximation, $c_0 = c_2 = 0$. Then follows that norm of error function is

$$I(a_1, a_3) = ||\delta_3||_2^2 = \int\limits_{-\pi}^{\pi} (\sin x - a_1 x - a_3 x^3)^2 \; dx,$$

$$\frac{\partial I}{\partial a_1} = -2 \int\limits_{-\pi}^{\pi} x(\sin x - a_1 x - a_3 x^3) \; dx = 0,$$

$$\frac{\partial I}{\partial a_3} = -2 \int\limits_{-\pi}^{\pi} x^3(\sin x - a_1 x - a_3 x^3) \; dx = 0.$$

Accomplish the task using integration by part, taking in account

$$\frac{1}{2} \int\limits_{-\pi}^{\pi} \sin x \; dx = \int\limits_{0}^{\pi} x \sin x \; dx = \pi,$$

$$\frac{1}{2} \int\limits_{-\pi}^{\pi} x^3 \sin x \; dx = \int\limits_{0}^{\pi} x^3 \sin x \; dx = \pi^3 - 6\pi.$$

[**3**] Using least-square (discrete) method determine parameters a_0 and a_1 in approximate function $\Phi(x) = a_0 + a_1 x$ for the following set of data:

j	0	1	2	3
x_j	0	1	2	4
$f(x_j)$	1	3	0	-1

Hint: Use the matrix equation

$$\mathbf{X^T} \cdot \mathbf{X} \cdot \vec{\mathbf{a}} = \mathbf{X^T} \cdot \vec{\mathbf{f}},$$

where, if we use the basic functions $\phi_i(x) = x^i \quad (i = 0, 1, \ldots, n)$,

$$\mathbf{X} = \begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & & & \\ 1 & x_m & x_m^2 & \ldots & x_m^n \end{bmatrix}, \quad (m \geq n).$$

Applying this, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 4 & -7 \\ -7 & 21 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5/7 \end{bmatrix}, \text{i.e.}$$

$$\Phi(x) = 2 - \frac{5}{7}x.$$

[4] Fit the equation of a straight line as well as possible (in a least squares sense) to the following data:

k	0	1	2	3	4
x_k	0	1	2	3	4
$f(x_k)$	1.00	3.85	6.50	9.35	12.05

Explore the general formula for least-square approximation over the discrete set of points.

Hint: See previous problem.

[5] By least-squares (discrete) method define parameters in approximation function $\Phi(x) = a_0 + a_1 x$ for the next data-set:

x	1	2	3	4
$f(x)$	1.95	2.40	2.83	3.30.

Repeat the procedure for quadratic approximation function $\Phi(x) = a_0 + a_1 x + a_2 x^2$.

[6] Obtain the approximation function of form $y = a \cdot e^{bx}$ for the next data:

x_j	1.0	1.5	2.	2.2
f_j	$e^{2.2}$	$e^{2.80}$	$e^{3.0}$	$e^{3.2}$.

Hint: Use the table

x_j	1	1.5	2.	2.2
$\log f_j$	2.2	2.80	3.0	3.2,

and after finding $\phi(x)$, replace by $y \cong e^{\phi(x)}$.

[7] Suppose that the following empirical data are available:

x	1.36	1.49	1.73	1.81	1.95	2.16	2.28	2.48
$f(x)$	14.094	15.069	16.844	17.378	18.435	19.949	20.963	22.495

Determine least-squares polynomial approximations $y_1(x)$ and $y_2(x)$ of degrees 1 and 2, respectively, weighting all data equally.

[8] The equations

$$\begin{array}{ccccccc} 2.17x_1 & + & 0.86x_2 & + & 1.17x_3 & = & 3.85 \\ 1.06x_1 & + & 2.81x_2 & - & 1.21x_3 & = & 3.03 \\ 1.91x_1 & - & 1.02x_2 & + & 3.91x_3 & = & 4.85 \\ 1.07x_1 & + & 1.21x_2 & + & 1.06x_3 & = & 3.27 \end{array}$$

are based on empirical data. Use the method of least squares to obtain approximate values of $x_1, x_2,$ and x_3.

[9] Proceed the discrete least squares approximation on the data set

x_i	1.1	1.9	4.2	6.1
$f(x_i)$	2.5	3.2	4.5	6.0

with functions

a) $\Phi_0(x) = a_0 + a_1 x,$

b) $\Phi_0(x) = a_0 + a_1 x + a_2 x^2.$

Develop the scalar and matrix formulas separately. Write a procedures in FORTRAN/Pascal/C. Compare the results with ones obtained by `Mathematica`.

[10] The experiments in some periodical process gave the following results:

t_j	$0°$	$50°$	$100°$	$150°$	$200°$	$250°$	$300°$	$350°$
f_j	0.754	1.762	2.041	1.412	0.303	−0.484	−0.380	0.520

Determine parameters a and b in model $\phi(x) = a + b \cdot \sin t$ by least-squares polynomial approximation.

Hint: Minimize the function

$$f(a,b) = \sum_{j=0}^{7} (f_j - a - b \sin t_j)^2.$$

One needs the following sums:

$$\sum_{k=0}^{7} \sin t_k, \ \sum_{k=0}^{7} (\sin t_k)^2, \ \sum_{k=0}^{7} f_k, \ \sum_{k=0}^{7} f_k^2, \ \sum_{k=0}^{7} f_k \sin t_k,$$

with solution $a \cong 0.75257, \ b \cong 1.31281$.

[11] Construct the approximative polynomial $\Phi(x) = a_0 + a_1 x + a_2 x^2$ by least-squares method, for the data-set:

x_j	−2	−1	0	1	2
$f(x)$	−0.1	0.1	0.4	0.9	1.6

[12] Using basic functions $\phi_0(x) = 1, \ \phi_1(x) = x-2, \ \phi_3(x) = x^2-4x+2$, and least-squares method, approximate data set

$$\{(0,-2),(1,2),(2,5),(3,3),(4,1)\}$$

by function

$$\phi(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x).$$

[**13**] The equation of form

$$\mathbf{A} = \begin{vmatrix} P(x) & 1 & x & x^2 & \dots & x^n \\ y_0 & 1 & x_0 & x_0^2 & \dots & x_0^n \\ y_1 & 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y_n & 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = 0$$

gives as a solution the interpolation polynomial $P(x)$. Write a procedure in system `Mathematica` for
a) analytic solution,
b) discrete solution, with given data-set.

[**14**] Write a procedure for realization of Lagrange's interpolation polynomial for evaluating a polynomial, for cases:
a) The whole polynomial form (all coefficients),
b) discrete value of polynomial for desired fixed value.

Use Fortran/Pascal/C and compare the results with ones obtained by `Mathematica`.

Data example: Make interpolation of function of normal distribution $y(x) = e^{-x^2/2}/\sqrt{2\pi}$ by Lagrange's formula and using it find $y(1.5)$, when given data-set

x_k	1.00	1.2	1.4	1.6	1.8	2.0
y_k	0.2420	0.1942	0.1497	0.1109	0.0790	0.0540

(Result: $y(1.5) = 0.1295$).

[**15**] The average results of golf competitions for different games are the following:

Game	6	8	10	12	14	16	18	20	22	24
Av.pts.	3.8	3.7	4.0	3.9	4.3	4.2	4.2	4.4	4.5	4.5

Make a linear approximation for given data-set using least-squares formula. Make a table with smoothed values of function.

Write a program in Fortran/Pascal/C and draw a graphic of rough data with approximation line.

[9] Proceed the discrete least squares approximation on the data set

x_i	1.1	1.9	4.2	6.1
$f(x_i)$	2.5	3.2	4.5	6.0

with functions

[16] Apply the approximation function of form $P = c \cdot e^{dx}$ to the data set

x_i	1	2	3	4
p_i	7	11	17	27

Hint: Taking $y = \log P = \log c + d \cdot x = a + b \cdot x$, we get the table

x_i	1	2	3	4
p_i	1.95	2.40	2.83	3.30

and the result $a \cong 1.5$, $b \cong 0.45$, i.e. $P = e^{a+bx} = e^a \cdot e^{bx}$. It is to mention that the approximation has been realized by minimization of $\sum[y(x_i) - y_i]^2$, and not by $\sum[P(x_i) - P_i]^2$, what is usual procedure.

Write a code in `Mathematica` and draw a graphic with rough data and approximation function.

[17] Make a parabolic function approximation for data given in problem no. 15. Use the obtained results for data smoothing. Write a program in `Mathematica`. Compare the results with linear approximation function. Draw a graphic of rough data with approximation curves.

[18] For the data set

x_i	2.2	2.7	3.5	4.1
p_i	65	60	53	50,

by applying $y = \log P$, construct the linear approximation function by least-squares method. (Solution: $P = 91.90 \cdot x^{-0.43}$.

[19] Obtain the approximative function of form $P = a \cdot e^{bx}$ for data set

x_i	1	2	3	4
p_i	60	30	20	15,

by substitution $y_i = \log P_i$.

[20] The following table contains the values of $y(x) = x^2$ with occasional error from -0.10 to 0.10. The exact values T_i are given in third row.

x_i	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y_i	0.98	1.23	1.40	1.72	1.86	2.17	2.55	2.82	3.28	3.54	3.92
T_i	1.00	1.21	1.44	1.69	1.96	2.25	2.56	2.89	3.24	3.61	4.00

Construct the quadratic (parabolic) approximation function and smooth the data in the table. Compare the original and obtained data. Compute average error in both cases. Write `Mathematica` and Fortran code for solving a problem.

[21] The following data contain the values of $\sin x$ with occasional error between -0.10 and 0.10. Construct the parabolic approximation function by least-square method and apply it to smooth the data.

x	0.	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$\sin x$	-0.09	0.13	0.44	0.57	0.64	0.82	0.97	0.98	1.04

[22]* Show how Gram-Schmidt orthogonalization can be used to solve $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ by orthogonalizing the columns of $\mathbf{A}$ and then using the method for orthogonal matrices.

[23]* For the four linear algebra problems

a. Solve $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$,
b. Compute $\mathbf{A^{-1}}$,
c. Find the least-squares solution to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$,
d. Compute $\det \mathbf{A}$,

discuss in a comparative manner the following points for noted problems:

1. The difficulty of the matrix theory underlying the computation;
2. The actual amount of computation required to solve the problems (i.e. the amount of execution time for comparable-sized matrices);
3. The amount of preliminary analysis that one would expect to make in order to write a library subroutine to solve each of these problems for a large class of matrices $\mathbf{A}$.

[24]* Write a Fortran/Pascal/C/Mathematica subroutine to carry out Gram-Schmidt orthogonalization and apply it to solve least-square problems. The specific examples to be used are:

a. $a_i = [\frac{1}{i+1}, \frac{1}{i+2}, \ldots, \frac{1}{i+n}]^T$ for $i = 1$ to 10 and $n = 20$,
$b_j = 1$ for all j.
b. $a_i = [\sin(\frac{i}{n}), \sin(\frac{2i}{n}), \sin(\frac{3i}{n}), \ldots, \sin(i)]^T$, for $i = 1, \ldots, 10$ and $n = 10$,
$b_1 = 1$ and $b_j = 0$ for $j > 1$.
c. a_i = random vector of lenght n for $i = 1$ to N,
b_j = random number.