Faculty of Civil Engineering Belgrade
Master Study COMPUTATIONAL ENGINEERING
Fall semester 2005/2006

ASSIGNMENTS

LESSON VII

Interpolation of Functions

[**1**] Determine approximately $f(1)$ using data from the table

k	0	1	2	3
x_k	-1	0	2	3
$f(x_k)$	-3	1	3	13

by Aitken's scheme.
Hint: Aitken's scheme is of form:

$$\mathbf{A}_k = f(x_k) \quad (k = 0, 1, \dots, n),$$

$$\mathbf{A}_{k-1,k} = \frac{1}{x_k - x_{k-1}} \begin{vmatrix} \mathbf{A}_{k-1} & x_{k-1} - x \\ \mathbf{A}_k & x_k - x \end{vmatrix} \qquad (k = 1, \dots, n),$$

$$\mathbf{A}_{0,1,\dots,n} = \frac{1}{x_n - x_0} \begin{vmatrix} \mathbf{A}_{0,1,\dots,n-1} & x_0 - x \\ \mathbf{A}_{1,2,\dots,n} & x_n - x \end{vmatrix},$$

where $P_n(x) = \mathbf{A}_{0,1,\dots,n}$. Following this procedure, one gets

$$\mathbf{A}_{0,1} = \frac{1}{0-(-1)} \begin{vmatrix} -3 & -1-1 \\ 1 & 0-1 \end{vmatrix} = 5,$$

$$\mathbf{A}_{1,2} = \frac{1}{2-0} \begin{vmatrix} 3 & 0-1 \\ 1 & 2-1 \end{vmatrix} = 2,$$

$$\mathbf{A}_{2,3} = \frac{1}{3-2} \begin{vmatrix} 3 & 2-1 \\ 13 & 3-1 \end{vmatrix} = -7,$$

$$\mathbf{A}_{0,1,2} = \frac{1}{2-(-1)} \begin{vmatrix} 5 & -1-1 \\ 2 & 2-1 \end{vmatrix} = 3,$$

$$\mathbf{A}_{1,2,3} = \frac{1}{3-0} \begin{vmatrix} 2 & 0-1 \\ -7 & 3-1 \end{vmatrix} = -1,$$

$$\mathbf{A}_{0,1,2,3} = \frac{1}{3-(-1)} \begin{vmatrix} 3 & -1-1 \\ -1 & 3-1 \end{vmatrix} = 1.$$

[2] Approximate function $x \mapsto f(x) = e^x$ on interval $[0, 0.5]$ by interpolating polynomial (Lagrange and Newton's interpolation formulas).
Hint: Function e^x given in tabular form

k	0	1	2
x_k	0.0	0.2	0.5
$f(x_k)$	1.00000	1.221403	1.648721

The approximative polynomial is of form $P_n(x) = a_0 x^n + a_1 x_{n-1} + \cdots + a_n$, given in Lagrange's form

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x),$$

$$L_k = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i},$$

i.e.

$$P_n(x) = \sum_{k=0}^{n} f(x_k) \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x-x_i}{x_k-x_i}.$$

By applying the data from the table, one gets

$$P_2(x) = 1.\frac{(x-0.2)(x-0.5)}{(0-0.2)(0-0.5)} + 1.221403\frac{(x-0.)(x-0.5)}{(0.2-0.)(0.2-0.5)} + 1.648721\frac{(x-0.)(x-0.2)}{(0.5-0.)(0.5-0.2)}.$$
$$= 0.634757x^2 + 0.980064x + 1$$

In order to construct Newton interpolating polynomial, we use finite differences of order r by recursive relation (forward difference)

$$[x_0, x_1, \ldots, x_r; f] = \frac{x_1, \ldots, x_r; f] - [x_0, \ldots, x_{r-1}; f]}{x_r - x_0},$$

where $x; f] = f(x)$. Newton interpolation polynomial has a form

$$P_n(x) = f(x_o) + (x-x_0)[x_0, x_1; f] + (x-x_0)(x-x_1)[x_0, x_1, x_2; f] + \ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1}[x_0, x_1, \ldots, x_n; f].$$

Table of divided differences is

k	x_k	f_k	Δf_k	$\Delta^2 f_k$
0	0.0	<u>1.000000</u>		
			<u>1.107150</u>	
1	0.2	1.221403		<u>0.634306</u>
			1.424303	
2	0.5	1.648721		

and Newton's polynomial has a following form.

$$P_n(x) = 1 + 1.107150x + (x-0)(x-0.2)0.634306$$
$$= 0.634306x^2 + 0.980288x + 1.$$

[3] For function $x \mapsto f(x) = \log x$ given in tabular form

k	0	1	2	3
x_k	0.40	0.50	0.70	0.80
$f(x_k)$	−0.916291	−0.693147	−0.356675	−0.223144

using Lagrange interpolation find approximately $\log 0.6$.

[4] For function given in previous example find the interpolating formula using

a) Newton's first and second interpolation function (forward and backward difference),
b) Gauss' first and second formula,
c) Bessel's formula, and
d) Stirling's formula.

[5] Using first Newton interpolating polynomial compute $\sin 6°$ based on the values of $\sin 5°, \sin 7°, \sin 9°$, and $\sin 11°$. Check if the same result is obtained by second Newton polynomial. *Hint:*

k	x_k	f_k	Δf_k	$\Delta^2 f_k$	$\Delta^3 f_k$
0	5°	0.087156			
1	7°	0.121869			
2	9°	0.156434			
3	11°	0.190809			

Use formula

$$P_n(x) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}\Delta^n f_0,$$

where $p = \frac{x-x_0}{n}$, or

$$P_n(x) = f_0 + \frac{\Delta f_0}{n}(x-x_0) + \frac{\Delta^2 f_0}{2!n^2}(x-x_0)(x-x_1) + \cdots + \frac{\Delta^n f_0}{n!n^n}(x-x_0)\ldots(x-x_{n-1}).$$

[6] Calculate $f(0.95)$ based on the following table of function values:

k	x_k	f_k
−2	0.5	−0.6875
−1	0.7	−0.8299
0	0.9	−0.9739
1	1.1	−0.9659
2	1.3	−0.6139

using first Gauss, second Gauss, and Stirling interpolation formulas. (6.1.15)

[7] Using Bessel interpolation formula get $\cos 14°$, based on values $\cos 11°, \cos 13°, \cos 15°$, and $\cos 17°$.

k	x_k	f_k
0	11°	0.98163
1	13°	0.97437
2	15°	0.96593
3	17°	0.95630

[8] Form an Hermite interpolation polynomial based on the function values given in the table

x	-1	0	2
$f(x)$	0	-7	3
$f'(x)$	-8	-5	55
$f''(x)$		10	0

Hint: (6.1.21).

[9] Based on table

x	$-\pi$	$-2\pi/3$	$-\pi/2$	0	$\pi/2$
$f(x)$	2	0.5	0	2	0

determine the trigonometrical interpolation polynomial.

Hint: Trigonometrical interpolation polynomial has a form (one of representative forms):

$$T_n(x) = \sum_{k=0}^{2n} f(x_k) \prod_{\substack{j=0\\ j\neq k}}^{n} \frac{\sin \frac{x-x_j}{2}}{\sin \frac{x_k-x_j}{2}}.$$

[10] Calculate approximate values of $f(x) = \sin x$ for $x = 0.50(0.02)0.70$ and for $x = 1.50(0.02)1.70$ by applying the approximate newtonian formula to the following rounded data:

k	x_k	f_k
0	0.5	0.47943
1	0.7	0.64422
2	0.9	0.78333
3	1.1	0.89121
4	1.3	0.96356
5	1.5	0.99749
6	1.7	0.99166

[11] Calculate approximate values of $f(1.0)$ from the data of Problem 10, first by use of Gauss' forward formula launched from $x = 0.9$ and second by use of the backward formula launched from $x = 1.1$.

[12] Use Stirling's formula to calculate approximate values of $f(x)$ for the points $x = 1.00(0.02)1.20$ from the data of Problem 10.

[13] Use Bessel's formula to calculate approximate values of $f(x)$ for the points $x = 0.90(0.02)1.10$ from the data of Problem 10.

[14] Derive the Newton's interpolation polynomial of third degree for given data

k	x_k	f_k
0	4	1
1	6	3
2	8	8
3	10	20.

Hint: Use Newton's forward formula

$$P(x_k) = f_0 + \frac{\Delta f_0}{n}(x_k - x_0) + \frac{\Delta^2 f_0}{2!n^2}(x_k - x_0)(x_k - x_1) + \cdots + \frac{\Delta^n f_0}{n!n^n}(x_k - x_0)\ldots(x_k - x_{n-1}).$$

Solution:

$$P(x_k) = \frac{1}{24}[2x_k^3 - 27x_k^2 + 142x_k - 240].$$

[**15**] Derive the Newton's interpolation polynomial of fourth or smaller degree, from the following data:

k	x_k	f_k
0	1	1
1	2	−1
2	3	1
3	4	−1
4	5	1

[**16**] Derive the polynomial of degree n, according to following tables of data:

a)

k	x_k	f_k	$n = 4$
0	2	0	
1	4	0	
2	6	1	
3	8	0	
4	10	0	

b)

k	x_k	f_k	$n = 2$
0	0	1	
1	1	2	
2	2	4	
3	3	7	
4	4	11	
5	5	16	
6	6	22	
7	7	29	

c)

k	x_k	f_k	$n=3$
0	3	6	
1	4	24	
2	5	60	
3	6	120	

d)

k	x_k	f_k	$n=5$
0	0	0	
1	1	0	
2	2	1	
3	3	1	
4	4	0	
5	5	0	

[**17**] Derive the polynomial of third degree which interpolates function $y(x) = \sin\left(\frac{\pi x}{2}\right)$ for $x = 0, 1, 2, 3$. Compare the values of original functions and interpolated for $x = 4$ and $x = 5$.

[**18**] Derive the polynomial of fourth degree which interpolates function $y(x) = \sin\left(\frac{\pi x}{2}\right)$ for $x = 0, 1, 2, 3, 4$.

[**19**] Derive the polynomial of second degree which interpolates function $y(x) = x^3$ for $x = -1, 0, 1$.

[**20**] Derive the polynomial of third degree using Newton's second interpolation formula, from data given in the table:

k	x_k	f_k
-3	4	1
-2	6	3
-1	8	8
0	10	20

[21] Apply the Gauss interpolation formula for obtaining the polynomial of fourth degree, taking data from the table:

k	x_k	y_k
-2	2	-2
-1	4	1
0	6	3
1	8	8
2	10	20

[22] Apply the Gauss interpolation formula with forward differences to obtain the Gauss' interpolation polynomial of degree four, from following data set:

k	x_k	y_k
-2	1	1
-1	2	-1
0	3	1
1	4	-1
2	5	1

Solution:

$$P_k = \frac{1}{3}(2k^4 - 8k^2 + 3), k = x_k - 3,$$
$$P_k = \frac{1}{3}(2x_k^4 - 24x_k^3 + 100x_k^2 - 168x_k + 93).$$

[23] Apply the Bessel interpolation formula for obtaining the polynomial of degree three, taking data from the following table

k	x_k	f_k	Δf_k	$\Delta^2 f_k$	$\Delta^3 f_k$
-1	4	1			
			2		
0	6	$\underline{3}$		$\underline{3}$	
			$\underline{5}$		$\underline{4}$
1	8	$\underline{8}$		$\underline{7}$	
			12		
2	10	20			

Hint (7.26): By taking Bessel's formula of form

$$P_k = \mu y_{1/2} + (k - \frac{1}{2})\delta y_{1/2} + \binom{k}{2}\mu\delta^2 y_{1/2} + (\frac{1}{3})(k - \frac{1}{2})\binom{k}{2}\delta^3 y_{1/2} + \dots$$
$$+ \binom{k+n-1}{2n}\mu\delta^{2n} y_{1/2} + \frac{1}{2n+1}(k - 1/2)\binom{k+n-1}{2n}\delta^{2n+1} y_{1/2},$$

where δ and μ are operators of central difference and middle value, respectively, one gets

$$P_k = \frac{3+8}{2} + 5(k - 1/2) + \frac{3+7}{2}\frac{k(k-1)}{2} + \frac{14}{3}(k - \frac{1}{2})\frac{k(k-1)}{2}.$$

Finally, we involve the relation $k = \frac{x_k}{2} - 3$ in previous formula.

[**24**] Apply the Newton's backwards difference interpolation formula in order to construct the polynomial of degree four using the following data:

k	x_k	y_k
0	5	1
-1	4	-1
-2	3	1
-3	2	-1
-4	1	1

Hint: Apply

$$P_k = y_0 + k\Delta y_0 + \frac{k(k+1)}{2!}\Delta^2 y_0 + \cdots + \frac{k(k+1)\cdots(k+n-1)}{n!}\Delta^n y_0,$$

and then replace $k = x_k - 5$.

[25] Apply the Gauss' formula with forward differences to the following data

k	x_k	y_k
-2	2	0
-1	4	0
0	6	1
1	8	0
2	10	0

Hint: Take

$$P_k = \sum_{i=0}^{n}\left[\binom{k+i-1}{2i}\delta^{2i}y_0 + \binom{k+1}{2i+1}\delta^{2i+1}y_{1/2}\right], \quad (k = \frac{x_k}{2} - 3).$$

For even degrees, $2n$, take nodes $k = -n, \ldots, n$. For odd degrees, $2n$, take nodes $k = -n, \ldots, n+1$.

[26] Apply the Gauss' backward formula for data given in previous problem.
Hint: Take

$$P_k = y_0 + \sum_{i=1}^{n}\left[\binom{k+i-1}{2i-1}\delta^{2i-1}y_{-1/2} + \binom{k+i}{2i}\delta^{2i}y_0\right],$$

for nodes $k = -n, \ldots, n$. Finally, replace k with $\frac{x_k}{2} - 3$.

[27] Apply the Gauss' formula with backward differences to the following data in table form

k	x_k	y_k
-2	0	0
-1	1	0
0	2	1
1	3	1
2	4	0
3	5	0

Hint: Apply the formula given in previous problem.

[**28**] Consider the following formulas and write the code in `Mathematica` for their realization.

<u>Stirling</u>

$$P_k = y_0 + \binom{k}{1}\delta\mu y_0 + \frac{k}{2}\binom{k}{1}\delta^2 y_0 + \binom{k+1}{3}\delta^3\mu y_0 + \frac{k}{4}\binom{k+1}{3}\delta^4 y_0 + \cdots + \binom{k+n-1}{2n-1}\delta^{2n-1}\mu y_0 + \frac{k}{2n}\binom{k+n-1}{2n-1}\delta^{2n} y_0,$$

for nodes $k = -n, \ldots, n$.

<u>Everett</u>

$$P_k = \binom{k}{1}y_1 + \binom{k+1}{3}\delta^2 y_1 + \binom{k+2}{5}\delta^4 y_1 + \cdots + \binom{k+n}{2n+1}\delta^{2n} y_1 - \binom{k-1}{1}y_0 - \binom{k}{3}\delta^2 y_0 - \binom{k}{5}\delta^4 y_0 - \cdots \binom{k+n-1}{2n+1}\delta^{2n} y_0,$$

for nodes (even) numbers $k = -n, \ldots, n+1$.

<u>Bessel</u>

$$P_k = \mu y_{1/2} + (k - \frac{1}{2})\delta y_{1/2} + \binom{k}{2}\mu\delta^2 y_{1/2} + \frac{1}{3}(k - \frac{1}{2})\binom{k}{2}\delta^3 y_{1/2} + \cdots + \binom{k+n-1}{2n}\mu\delta^{2n} y_{1/2} + \frac{1}{[2n+1]}(k - \frac{1}{2})\binom{k+n-1}{2n}\delta^{2n+1} y_{1/2}.$$

[29] For given values $y(x) = \sqrt{x}$ form form a table of differences up to Δ^6. Apply the table to calculate $\sqrt{1.005}$ with $n = 1$ (linear approximation) using Newton's forward differences.

k	x_k	$y(x) = \sqrt{x}$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	1.0	1.0000						
			50					
1	1.01	1.0050		0				
			50		−1			
2	1.02	1.0100		−1		2		
			49		1		−3	
3	1.03	1.0149		0		−1		4
			49		0		1	
4	1.04	1.0198		0		0		
			49		0			
5	1.05	1.0247		0				
			49					
6	1.06	1.0296						

Hint: The Newton's formula is

$$P_k = y_0 + \binom{k}{1}\Delta y_0 + \frac{k}{2}\binom{k}{2}\Delta^2 y_0 + \cdots + \binom{k}{n}\Delta^n y_0,$$

so that for $n = 1$ we have

$$k = \frac{x - x_0}{h} = \frac{1.005 - 1.000}{0.01} = \frac{1}{2},$$

i.e. $P_k = 1.000 + \frac{1}{2}(0.0050) = 1.0025.$