Faculty of Civil Engineering Belgrade
Master Study COMPUTATIONAL ENGINEERING
Fall semester 2005/2006

ASSIGNMENTS

LESSON VI

Non-linear Equations

and Systems of Equations

1. Find the real root of equation

$$x^3 - x - 1 = 0$$

   by iteration method.

2. The function

$$x \to g(x) = \frac{x^3}{0.05 - \frac{e^{-x}}{1+x}}$$

   has a local minimum in $x = a \cong 2.5$. Find its value by iterative method with accuracy $\varepsilon = 10^{-3}$.

3. For function

$$f(x) = x^2 - ax(\log x - 1)$$

   there is one value $a = A$ so that $f'(x) = f''(x) = 0$, for some x. Determine A with $\varepsilon = 10^{-3}$.

4. Determine a square root of positive number a by Newton method with arbitrary accuracy.

5. Find a root of equation

$$f(x) = e^{-x} - x = 0$$

with Determine a square root of positive number a by Newton method with $\varepsilon = 5 \cdot 10^{-2}$. by bisection.

6. Solve the equation

$$f(x) = x^2 - e^x + 2 = 0$$

with $\varepsilon = 10^{-4}$. using secant method, and then regula-falsi method.

7. Suppose the equation $x^2 + a_1 x + a_2$ possesses real roots α and β. Show that the iteration

$$z_{k+1} = -(a_1 z_k + a_2)/z_k$$

is stable at $x = \alpha$ if $|\alpha| > |\beta|$, the iteration

$$z_{k+1} = \frac{-a_2}{z_k + a_1}$$

is stable at $x = \alpha$ if $|\alpha| < |\beta|$, and the iteration

$$z_{k+1} = -(z_k^2 + a_2)/a_1$$

is stable at $x = \alpha$ if $2|\alpha| < |\alpha + \beta|$.

8. Show that if the asymptotic convergence factor ρ of an iteration can be estimated in any way, then the formula

$$\alpha \approx z_{k+1} + \frac{\rho}{1-\rho}(z_{k+1} - z_k)$$

can be used to accelerate the convergence of the iteration in place of the Aitken Δ^2 process, and also that latter process is equivalent to estimating ρ by the ratio

$$\frac{z_{k+2} - z_{k+1}}{z_{k+1} - z_k}.$$

9. The real root α of the equation $x+\log x=0$ lies between 0.56 and 0.57. Show that the iteration $z_{k+1}=-\log z_k$ is unstable at $x=\alpha$, and verify this fact by calculation. Then show that the iteration $z_{k+1}=e^{-z_k}$ is stable at $x=\alpha$, and determine α to five places.

10. Solve the nonlinear equation

$$x+1=x^4$$

by fixed-point iteration.

Hint: Start from the interval $[0,3]$. Compare iteration formulas

$$x_{k+1}=x_k^4-1$$

and

$$x_{k+1}=\sqrt[4]{x_k+1}.$$

(In the second case the fixed point is attractive while it is repulsive in the first case).

11. Write a FORTRAN/Pascal/C subprogram to implement the bisection method, where $BISECT$ is the name of subroutine, F name of non-linear function $[A,B]$ is the interval containing a zero, and EPS is accuracy required in the zero. The value of the $BISECT$ is the zero found.

Apply the subprogram to compute zeros of the following functions:

a) $\sin(x-0.23450+0.5;$
b) $xe{-}x-0.2;$
c) $\log(1+x^2)-\frac{1}{x+1};$
d) $\cos x-\sqrt{|x|};$
e) $x^4-4x^2+2x-13;$
b) $x-(8x^3-12x^2)\log|10.5-x|-6;$

Choose a value of EPS between 10^{-2} and 10^{-5}. A and B determine graphically.

12. Write a FORTRAN/Pascal/C subprogram to implement the regula falsi method with name RF, F name of the function,$[A, B]$ is the interval containing a zero, and EPS is accuracy required in the zero.Apply the subprogram to compute zeros of the functions a) through f) of previous problem, and of
    a) $x^{10} - 0.01$(start with $A = 0, B = 1$;
    b) $\tan x - 0.05$(start with $A = 0, B = 1.55$;
    c) $10e^{-10x} - 1/over10$ (start with $A = 0, B = 2$.

13. Write a FORTRAN/Pascal/C subprogram to implement Newton's method named $NEWTON$, with arguments F-function name, FP- name of its derivative, $GUESS$-initial estimate of the zero, and EPS is the accuracy desired. The value of function is the zero found. Apply the subprogram to compute zeros of the following functions:
    a) x^2(guess $x = 1$);
    b) x^{10} (guess $x = 1$ or 2);
    c) $x^3 - 3x + 3x^2 - 1$ (guess $x = 0$ or 20);
    d) $\sqrt{|x - 1|}$(guess $x = 0$ or 2));
    e) $(x^2 + \frac{1}{x})e^{x+\sin x} \cos(x \log(1 + x^2)) - 0.1$ (guess $x = 0.01, -8$ or 12);
    f) $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$ (guess $x = 2$ or 20);
    g) $|\sin x - 0.2|$ (guess $x = 0.1$ or 1.57);
    h) $\sqrt[5]{x - 0.2)(x - 0.3)^2}$ (guess $x = 0$ or 0.25).

    Choose a value of EPS between 10^{-4} and 10^{-5}.

14. Write a FORTRAN/Pascal/C subprogram to implement the secant method named $SECANT$, and arguments $GUESS1$ and $GUESS2$, two guesses for the zero, and F-function name

and EPS-accuracy desired. Apply the subprogram to the functions given in previous problem.

15. Write a FORTRAN/Pascal/C subprogram to implement the fixed point method named $FIXPT$, where the arguments are F-function name, $GUESS$-initial value, EPS- desired accuracy. Apply the subprogram to find zeros of the functions:
   a) $x^2 - x$(guess $x = 0.1, 0.4, -0.2, -0.8, -1.5, 1.5$);
   b) $\sin x - 0.25$ (guess $x = 0.2$);
   c) $\cos x - 0.25$ (guess $x = 1.3, 10.75, -4.1, 732.7$);
   d) $x - e^x$(guess $x = 0., 0.75, 20$);
   e) $x + \sin x - \frac{x^3}{3}$(guess $x = 0.25$).
   Take EPS from 10^{-2} to 10^{-4}.

16. The problem "Solve $x^2 - x - 2 = 0$" can be reformulated in several ways for fixed point iteration, including

$$x = x^2 - 2,$$
$$x = \sqrt{x + 2},$$
$$x = 1 + \frac{2}{x},$$
$$x = x - \frac{x^2 - x - 2}{4}.$$

   Explore the convergence of given iteration process.
   For the following equations reformulate the fixed point iterations so that those converge:
   a) $x^3 - x + 1 = 0$;
   b) $e^x - \sin x = 0$;
   c) $\log(1 + x) - x^2 = 0$ (for both zeros);
   d) $e^x - 3x^2 = 0$ (for both zeros).

17. In the year 1225 explored Leonardo of Pisa the equation

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0$$

and got the root $x = 1.368808107$, using unknown method. This result was outstanding for that time.

Solve the equation of Leonardo by reduction to the form $x = F(x)$ (fixed point method).

Hint: Take

$$x = F(x) = \frac{20}{x^2 + 2x + 10}, \text{ i.e.}$$
$$x_n = \frac{20}{x_{n-1}^2 + 2x_{n-1} + 10}, \text{ and } x_0 = 1.$$

18. Apply the concept of extrapolation on the limit value in order to improve the iterative method (Aitken Δ^2 method). *Hint:* The error of computation is (after n iterations)

$$e_n = r - x_n = F(r) - F(x_{n-1}) = F'(\xi)(r - x_{n-1}) = F'(\xi)e_{k-1},$$

i.e.

$$e_n \approx F'(r)e_{n-1}.$$

Without knowing r and $F'(r)$, one can say

$$r - x_{n+1} \approx F'(r)(r - x_n)$$
$$r - x_{n+2} \approx F'(r)(r - x_{n+1}),$$

and get, by division

$$\frac{r - x_{n+1}}{r - x_{n+2}} \sim \frac{r - x_n}{r - x_{n+1}}.$$

Finally, one gets

$$r \approx x_{n+2} - \frac{(x_{n+2} - x_{n+1})^2}{x_{n+2} - 2x_{n+1} + x_n} = x_{n+2} - \frac{(\Delta x_{n+1})^2}{\Delta^2 x_n}.$$

Thus, if three successive iterations x_k, x_{k+1}, and x_{k+2} are known, this relation affords an *extrapolation* which may be expected to provide an improved estimate of r, when the iteration converges. This procedure for accelerating convergence is often called Aitken's Δ^2 process.

19. The method of Steffenson is the modification of iterative method by Aitken's method in every third iteration. Apply this procedure to Leonardo equation

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0.$$

Hint: One gets three first values taking $x_0 = 1$, by fixed point method, using formula

$$x_k = \frac{20}{x_{k-1}^2 + 2x_{k-1} + 10}, \qquad k = 1, 2$$

and the third value using Aitken's method

$$x_k = x_{k-1} - \frac{(x_{k-1} - x_{k-2})^2}{x_{k-1} - 2x_{k-2} + x_{k-3}} \qquad k = 3, 6, \ldots$$

This process is repeated until the wanted accuracy ε is achieved.

20. Apply the Newton method on the equation of Leonardo of Pisa

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0,$$

with accuracy $\varepsilon = 10^{-k} \quad (k = 1, 2, \ldots, 10)$. Compare the number of function and first derivative of function evaluations. Write a procedure in FORTRAN/Pascal/C. Compare with the results obtained by program Mathematica.

21. Prove that formula for obtaining a square root

$$x_n = \frac{1}{2}(x_{n-1} + \frac{Q}{x_{n-1}})$$

is a special case of Newton iteration formula.
Hint: Take $f(x) = x^2 - Q = 0$.

22. Derive the iterative formula for calculation of p–th root of Q

$$x_n = x_{n-1} - \frac{x_{n-1} - Q}{p \cdot x_{n-1}^{p-1}},$$

as a special case of Newton formula.

23. Apply regula falsi method to equation of Leonardo of Pisa

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0.$$

The starting values x_0 and x_1 are to be defined graphically.

24. Consider the system

$$\begin{aligned} 6x &= \cos x + 2y \\ 8y &= xy^2 + \sin x. \end{aligned}$$

Using fixed-point iterations

$$\begin{aligned} x_{k+1} &= \frac{1}{6}(\cos x_k + 2y_k) \\ y_{k+1} &= \frac{1}{8}(x_k y_k^2 + \sin x_k), \end{aligned}$$

with $x_0, y_0 \in [0, 1]$, solve the system.

25. Compute a solution of the equations

$$\begin{aligned} f_1(x, y) &:= x^3 - y^3 - x^2 y - 7 = 0 \\ f_2(x, y) &:= x^2 + y^2 - 4 = 0 \end{aligned}$$

using Newton's iteration, starting from the point on the circle with radius 2 as an initial guess, namely $(x_0, y_0) = (2, 0)$ and $f(x_0, y_0) = (1, 0)$.

Hint: Formal expression of Newton iteration is

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} - \mathbf{J}^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

i.e.

$$\mathbf{J} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where $\mathbf{J}$ Jacobi matrix (for two unknowns) of form

$$\mathbf{J} = \begin{bmatrix} \frac{\delta f_1}{\delta x} & \frac{\delta f_1}{\delta y} \\ \frac{\delta f_2}{\delta x} & \frac{\delta f_2}{\delta y} \end{bmatrix},$$

i.e. for given case

$$\mathbf{J} = \begin{bmatrix} 3x^2 - 2xy & -3y^2 - x^2 \\ 2x & 2y \end{bmatrix},$$

what gives the system of linear equations for the first Newton correction

$$\begin{bmatrix} 12 & -4 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This yields the new iterate

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{4} \end{bmatrix}, \quad f \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} -1/64 \\ 1/16 \end{bmatrix}.$$

These vectors lead to the new linear system

$$\begin{bmatrix} 11 & -4\frac{3}{16} \\ 4 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{16} \end{bmatrix}.$$

The result and the other quantities of the Newton iteration are found in the following table. The vectors in the fourth step are already so close to the solution that the residues heavily depend on the rounding errors.

k	x_k	y_k	$f(x_k)$	$f(y_k)$	Δx	Δy
0	2.0	0.0	1.0	0.0	0.0	0.25
1	2.0	0.25	$-1/64$	$1/16$	$-1.141 \cdot 10^{-2}$	$-3.371 \cdot 10^{-2}$
2	1.98859	0.2163	$-1.60 \cdot 10^{-3}$	$1.27 \cdot 10^{-3}$	$-2.135 \cdot 10^{-4}$	$-9.646 \cdot 10^{-4}$
3	1.98837498	0.21532749	$-1.16 \cdot 10^{-6}$	$9.76 \cdot 10^{-7}$	$-1.663 \cdot 10^{-7}$	$-7.306 \cdot 10^{-7}$
4	1.98837481	0.21532676	$< 10^{-11}$	$< 10^{-11}$	$< 10^{-12}$	$< 10^{-13}$